MATH 2250 Calculus I
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Name: Answers
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Practice Test No. 3

Show all of your work, label your answers clearly, and do not use a calculator.

Problem 1 State the following theorems:

a Rolle's Theorem

If $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b), and if $f(a)=f(b)$, then there exists at least one point $c \in (a,b)$ such that $f'(c)=0$.

b The Mean Value Theorem

If $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b), then there exists at least one point c such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

Problem 2

a Does the Mean Value Theorem apply to the function $f(x) = |x|$ on the interval $[-1, 1]$?

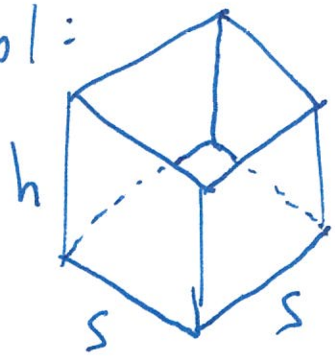
No, because $f'(x)$ does not exist when $x = 0$, so $f(x)$ is not differentiable on $(a, b) = (-1, 1)$.

b Does the Mean Value Theorem apply to the function $g(x) = 1/x$ on the interval $[0, 1]$?

No, because $g(x)$ is not defined at $x = 0$, so $g(x)$ is not continuous on $[a, b] = [-1, 1]$.

Problem 3 A box with a square base and an open top must have a volume of 32,000 cm^3. Find the dimensions of the box that minimizes the amount of material used.

Step 1:



Step 2: Objective function:

$$A(s, h) = s^2 + 4sh$$

Step 3: Constraint equation:

$$32000 = s^2 h$$

$$\Rightarrow h = \frac{32000}{s^2} \Rightarrow A(s) = s^2 + 4s\left(\frac{32000}{s^2}\right)$$
$$= s^2 + 4(32000)s^{-1}$$

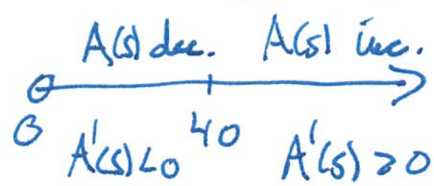
Step 4: Domain: We know $s \geq 0$, $h \geq 0$ because lengths. However, we also know that if either $s = 0$ or $h = 0$, $32000 \neq s^2 h$, so $s > 0$, $h > 0$. Note also that there is no upper bound on s. $\Rightarrow$ Domain: $(0, \infty)$

Step 5: Minimize $A(s)$ on $(0, \infty)$

$A'(s) = 2s - 4(32000)s^{-2}$, does not exist when $s = 0$, but that is outside the domain $(0, \infty)$. So set $A'(s) = 0$

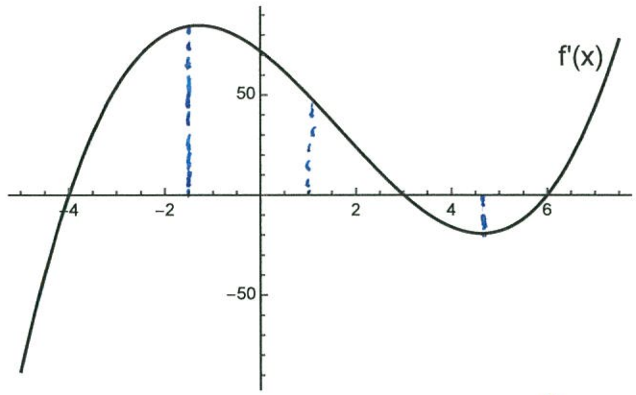
$$\Rightarrow 0 = 2s - 4(32000)s^{-2} \Rightarrow 2s = 4(32000)s^{-2}$$
$$\Rightarrow s^3 = 2(32000) \Rightarrow s = \sqrt[3]{64000} = \sqrt[3]{64}\sqrt[3]{1000} = 4(10)$$

$\Rightarrow s = 40$ is the only critical point. Check if minimum



Step 6: The dimensions of the box that minimize surface area are $s = 40$ cm and $h = 32000/(40^2)$ cm.

Problem 4 Below is a graph of the *derivative* of $f(x)$. This is a graph of $f'(x)$; do *not* make the mistake of thinking this is a graph of $f(x)$. Use this graph to answer the following questions about $f(x)$, $f'(x)$, and $f''(x)$.



a Where does the function $f(x)$ have critical points? $f'(x)$ always exists; $f'(x) = 0$ when the graph of $f'(x)$ crosses the x-axis $\Rightarrow x = -4, 3, 6$ are critical points.

b Where does the function $f'(x)$ have critical points?

Where $f'(x)$ has a horizontal tangent line $\Rightarrow x \approx -1.5, 4.25$

c Where does the function $f''(x)$ have critical points?

Where $f'(x)$ has a candidate for an inflection point, so $x = 1$

d Where is the function $f(x)$ increasing?

Where $f'(x) > 0$, so on $(-4, 3)$ and $(6, \infty)$

e Where is the function $f'(x)$ increasing?

On $(-\infty, -1.5)$ and $(4.25, \infty)$

f Where is the function $f''(x)$ increasing?

When $f'(x)$ is concave up, so on $(1, \infty)$

g Where does the function $f(x)$ have inflection points?

Where $f'(x)$ changes from inc. to dec. or dec. to inc., i.e. at local extrema of $f'(x)$, so $x = -1.5, 4.25$

h Where does the function $f'(x)$ have inflection points?

At $x = 1$.

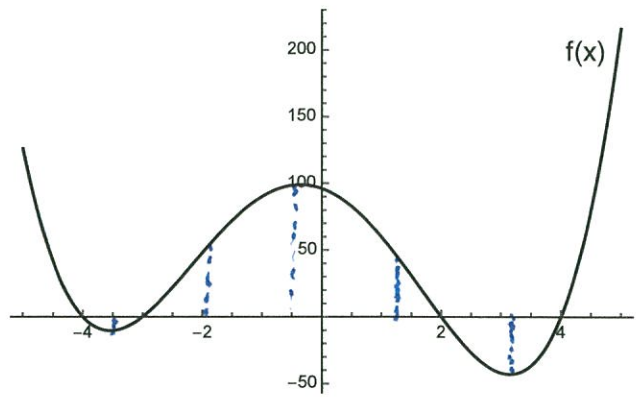
i Where is the function $f(x)$ concave up?

Where $f'(x)$ is increasing, so on $(-\infty, -1.5)$ and $(4.25, \infty)$

j Where is the function $f'(x)$ concave up?

On $(1, \infty)$.

Problem 5 Below is a graph of $f(x)$. Use this graph to answer the following questions about $f(x)$, $f'(x)$, and $f''(x)$.



a Where does the function $f(x)$ have critical points?

At $x \approx -3.5, -.5, 3.2$

b Where does the function $f'(x)$ have critical points?

Where f(x) has candidates for inflection points, so at $X = -2, 1.25$

c Where is the function $f(x)$ increasing?

On $(-3.5, -.5)$ and $(3.2, \infty)$

d Where is the function $f'(x)$ increasing?

Where f(x) is concave up, so on $(-\infty, -2)$ and $(1.25, \infty)$

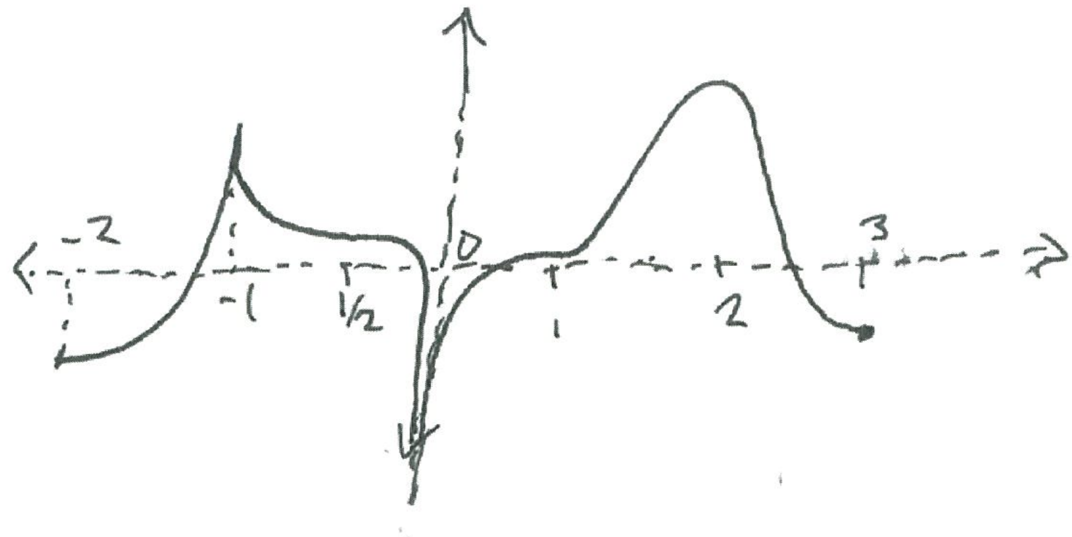
e Where does the function $f(x)$ have inflection points?

At $x = -2, 1.25$

f Where is the function $f(x)$ concave up?

On $(-\infty, -2)$ and $(1.25, \infty)$.

Problem 6 For the given graph of $f(x)$, answer the questions below.



a Where does the function $f(x)$ have points where $f'(x)$ does not exist?

$x = -1, 0,$

b Where does the function $f(x)$ have points where $f'(x) = 0$?

$x = 1/2, 1, 2$

c Where does the function $f(x)$ have relative maxima?

$x = -1, 2$

d Where does the function $f(x)$ have relative minima?

$x = -2, 3$

e Where does the function $f(x)$ have global maxima?

$x = 2$

f Where does the function $f(x)$ have global minima?

Does not exist.

Problem 7

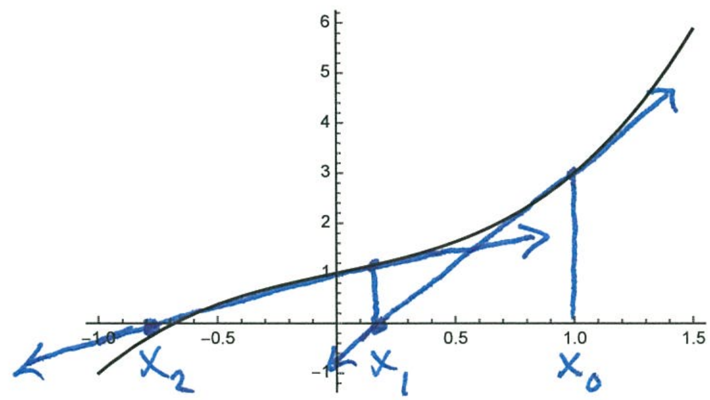
a Use the function $f(x) = x^3 + x + 1$ and the starting point $x_0 = 1$ to run two iterations of Newton's method, i.e. find x_1 and x_2. (You do not have to simplify x_2).

$$f'(x) = 3x^2 + 1$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{3}{4} = \frac{1}{4}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{1}{4} - \frac{\left(\frac{1}{4}\right)^3 + \frac{1}{4} + 1}{3\left(\frac{1}{4}\right)^2 + 1}$$

b On the graph of $f(x)$ below, show graphically what Newton's method is doing. (You should be drawing some lines on the graph).



Problem 8 Evaluate the following limits, making sure that if you use L'Hôpital's rule you have written sufficient justification:

a $\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)$ $= \lim_{x\to\infty} \frac{\sin(\frac{1}{x})}{1/x}$ yields $\frac{0}{0}$, so we can use L'Hôpital's rule $\Rightarrow \lim_{x\to\infty} \frac{\sin(\frac{1}{x})}{1/x} = \lim_{x\to\infty} \frac{\cos(\frac{1}{x})(\frac{-1}{x^2})}{(\frac{-1}{x^2})}$

$= \lim_{x\to\infty} \cos\left(\frac{1}{x}\right) = \cos\left(\lim_{x\to\infty} \frac{1}{x}\right)$ because $\cos(x)$ is continuous.

$= \cos(0) = 1$

b $\lim_{x\to 0} \frac{3x^2}{\cos(x) - 1}$ yields $\frac{0}{0}$, so use L.H.

$\Rightarrow \lim_{x\to 0} \frac{3x^2}{\cos(x)-1} = \lim_{x\to 0} \frac{6x}{-\sin(x)}$ yields $\frac{0}{0}$, so use L.H. again

$\Rightarrow = \lim_{x\to 0} \frac{6}{-\cos(x)} = -6$

c $\lim_{x\to 0^+} x^x$

Need to rewrite using $e^{\log(x)}$: $\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{\log(x^x)}$

$= \lim_{x\to 0^+} e^{x\log(x)} = e^{\lim_{x\to 0^+}\left(\frac{\log(x)}{1/x}\right)}$ because e^t is continuous.

and $\lim_{x\to 0^+} \frac{\log(x)}{1/x}$ yields $\frac{-\infty}{\infty}$, so can use L.H.

$\Rightarrow \lim_{x\to 0^+} x^x = e^{\lim_{x\to 0^+}\left(\frac{1/x}{-1/x^2}\right)} = e^{\lim_{x\to 0^+}(-x)} = e^0 = 1$

Problem 9 Find the global maximum and global minimum of the function given by $f(x) = -(x+1)^{2/3} + 3$ on the interval $[-2, 2]$ and where each of these values occur.

Find critical points: $f'(x) = -\frac{2}{3}(x+1)^{-1/3} = \dfrac{-2}{3\sqrt[3]{x+1}}$

DNE when $x = -1$, never equals zero.

So check critical points and end points

x	-2	-1	2
$f(x)$	2	3	$-\sqrt[3]{9}+3 \approx 1$

because $\sqrt[3]{9} > \sqrt[3]{8} = 2$

$\Rightarrow 3 - \sqrt[3]{9} < 1$

So the global max is 3 and occurs at $x = -1$ and the global min is $3 - \sqrt[3]{9}$ and occurs at $x = 2$.

Problem 10 Given the function $f(x) = (x+7)(x+4)(x+3)x$ on the interval $[-7, 0]$, answer the following questions:

a What are the x-intercepts of $f(x)$?

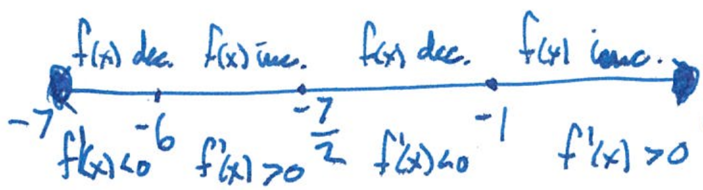
$x = -7, -4, -3, 0$

b I'll go ahead and tell you that the derivative is $f'(x) = 2(x+1)(x+6)(2x+7)$. What are the critical points of $f(x)$?

$x = -1, -6, \frac{-7}{2} = -3.5$

c On which intervals is $f(x)$ increasing?

Where $f'(x) > 0$, so on $(-6, \frac{-7}{2})$ and $(-1, 0)$



d Classify the critical points of $f(x)$ as local maxima, local minima, or neither.

$x = -1, -6$ are local minima, because f(x) changes from dec. to inc. there. $x = \frac{-7}{2}$ is a local max, because the opposite.

Correct this

e What is ~~are~~ the global maximum~~(s)~~ and the global minimum~~s~~ of $f(x)$? ~~(A function can have more than one global maximum or minimum if f attains the same value at those points.)~~

x	-7	-6	$\frac{-7}{2}$	-1	0
$f(x)$	0				0

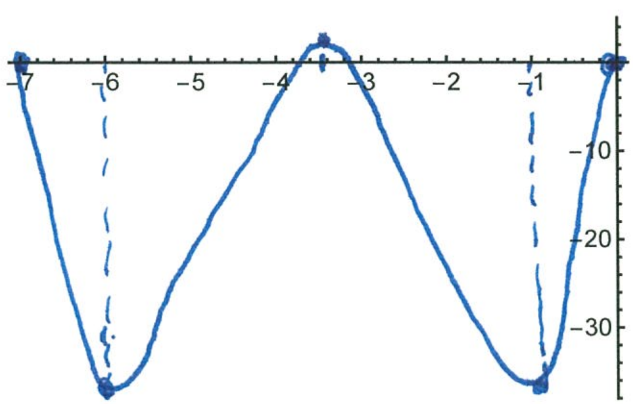
← endpoints and critical points.

$f(-6) = (1)(-2)(-3)(-6)$
$= -36$

$f(\frac{-7}{2}) = (\frac{7}{2})(\frac{1}{2})(\frac{-1}{2})(\frac{-7}{2})$
$= \frac{49}{16} \approx 3$

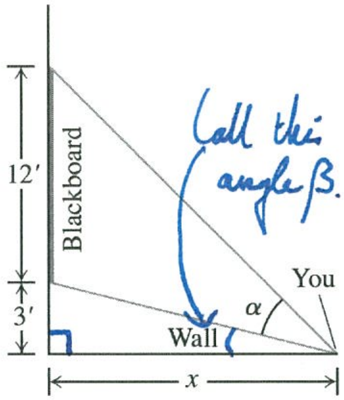
$f(-1) = (6)(3)(2)(-1)$
$= -36$

f Sketch the graph of $f(x)$. (Don't worry about concavity for this problem).



Problem 11 You are sitting in a classroom next to the wall looking at the blackboard at the front of the room. The blackboard is 12 ft long and starts 3 ft from the wall you are sitting next to. What is the maximum viewing angle, α?

Step 1: Picture



Call this angle β.

Step Two: Objective function : $\alpha(x)$

We want to maximize α, which is a function of x, the distance from the wall.

Step 3: We have no constraint, but we do need to solve for what $\alpha(x)$ is.

Using the angle β in the picture, we know two things:

$$\tan(\beta) = \frac{3}{x} \quad \text{and} \quad \tan(\alpha+\beta) = \frac{12+3}{x}$$

$$\Rightarrow \beta = \arctan\left(\frac{3}{x}\right) \quad \text{and} \quad \alpha+\beta = \arctan\left(\frac{15}{x}\right)$$

$$\Longrightarrow \alpha(x) = \arctan\left(\frac{15}{x}\right) - \arctan\left(\frac{3}{x}\right)$$

Step 4: Domain : $[0, \infty)$

Step 5: Maximize $\alpha(x)$ on $[0, \infty)$

$$\alpha'(x) = \left(\frac{1}{1+\left(\frac{15}{x}\right)^2}\right)\left(\frac{-15}{x^2}\right) - \left(\frac{1}{1+\left(\frac{3}{x}\right)^2}\right)\left(\frac{-3}{x^2}\right) = \frac{-15}{x^2+15^2} + \frac{3}{x^2+3^2}$$

always exists, so set $\alpha'(x) = 0 \Rightarrow \dfrac{3}{x^2+9} = \dfrac{15}{x^2+225}$

$$\Rightarrow 3x^2 + 3(225) = 15x^2 + 9(15) \Rightarrow 0 = 12x^2 + 9(15) - 3(15)^2$$

$$\Rightarrow x = 3\sqrt{5}$$

$0 \quad \alpha'(x) > 0 \quad 3\sqrt{5} \quad \alpha'(x) < 0$

Step 6: The max viewing angle is $\arctan\left(\frac{15}{3\sqrt{5}}\right) - \arctan\left(\frac{3}{3\sqrt{5}}\right)$ radians.